

# BOUNDS ON TRANSVERSE MOMENTUM DEPENDENT DISTRIBUTION FUNCTIONS

A. HENNEMAN

*Division of Physics and Astronomy, Faculty of Science, Vrije Universiteit  
De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands*

Received XXX

When more than one hadron takes part in a hard process, an extended set of quark distribution and fragmentation functions becomes relevant. In this talk, the derivation of Soffer-like bounds for these functions, in the case of a spin- $\frac{1}{2}$  target [1], is sketched and some of their aspects are discussed.

## 1 Introduction

In hard inclusive electro-weak processes the soft physics is described by light-cone correlators of quark fields. For a target hadron, for instance, all the relevant soft physics resides in the correlator [2, 3, 4]

$$\Phi_{ij}(x) = \int \frac{d\xi^-}{2\pi} e^{ip \cdot \xi} \langle P, S | \bar{\psi}_j(0) \psi_i(\xi) | P, S \rangle \Big|_{\xi^+ = \xi_T = 0}. \quad (1)$$

Here,  $P$  and  $S$  denote the parent hadron momentum and spin, and the relevant component of the quark momentum is  $x = p^+/P^+$ , the light-cone momentum fraction. The components  $a^\pm = a \cdot n_\mp$  stem from vectors  $n_+$  and  $n_-$ , satisfying  $n_+^2 = n_-^2 = 0$  and  $n_+ \cdot n_- = 1$ , which are fixed by the momentum that introduces the large scale  $Q$ , together with a (soft) hadron momentum. When only the leading part in orders of  $1/Q$  is considered, just the  $\Phi\gamma^+$  part of the correlator suffices. This part is usually parametrized in terms of the following quark distribution functions [5]

$$\Phi(x)\gamma^+ = \left\{ f_1(x) + S_L g_1(x) \gamma_5 + h_1(x) \gamma_5 \not{S}_T \right\} \mathcal{P}_+, \quad (2)$$

where  $\mathcal{P}_+$  stands for the projector of good fields  $\psi_+ \equiv \mathcal{P}_+ \psi = \frac{1}{2} \gamma^- \gamma^+ \psi$  [6]. For these functions some trivial bounds and the, less trivial, so-called Soffer bound have been derived [7].

If, now, one regards processes involving more than one hadron [8, 9], quark transverse momentum becomes relevant [10]. A correlator with transverse momentum leads to an extended set of distribution functions [11]. The purpose of this talk is to sketch the derivation of bounds for the additional functions in this extended set.

## 2 Light-front densities

A first observation in the derivation of the bounds is that the leading part of the correlator, now being non-diagonal in target spin space in contrast to (2),

$$(\Phi\gamma^+)_{ij,s's} = \int \frac{d\xi^-}{2\pi\sqrt{2}} e^{ip\xi} \langle P, s' | \psi_{+j}^\dagger(0) \psi_{+i}(\xi) | P, s \rangle \Big|_{\xi^+ = \xi_T = 0}, \quad (3)$$

after inserting a complete set of intermediate states, can be written in the following way

$$(\Phi\gamma^+)_{ij,s's} = \frac{1}{\sqrt{2}} \sum_n \langle P_n | \psi_{+j}(0) | P, s' \rangle^* \langle P_n | \psi_{+i}(0) | P, s \rangle \delta(P_n^+ - (1-x)P^+), \quad (4)$$

which is a positive semi-definite quantity. This property is not affected by inclusion of transverse momentum.

Next, target spin dependence is incorporated using a spin density matrix formalism.

$$M(S) = \text{Tr} [\rho(P, S) \tilde{M}(P)] \quad (5)$$

All target polarization information is in  $\rho(P, S)$ , while the spin dependence resides in the higher dimensionality of  $\tilde{M}(P)$ . For a spin-1/2 target,  $S$  is just a vector with properties  $P \cdot S = 0$  and  $-1 \leq S^2 \leq 0$  (being equal to  $-1$  for a pure state) and  $\tilde{M}$  is just  $2 \times 2$  in target spin space. In the target rest frame  $\rho(P, S)$  simplifies to  $1 + \mathbf{S} \cdot \boldsymbol{\sigma}$  and  $\tilde{M}$  assumes the form

$$\tilde{M}_{ss'} = \begin{pmatrix} M_O + M_L & M_T^1 - i M_T^2 \\ M_T^1 + i M_T^2 & M_O - M_L \end{pmatrix} \quad (6)$$

where the subscripts refer to target polarization  $S = (0, \mathbf{S}_T, S_L)$ , where  $S_L = MS^+/P^+$ . From the diagonal elements of this matrix one sees that it lives in the space spanned by states with  $S_L = 1$  and  $S_L = -1$ . In order to describe transverse target polarization one needs the off-diagonal elements.

Now, we turn our attention to quark spin. The analogon of (2) when quark transverse momentum is taken into account, is given by the sum of three parts

$$\Phi_O(x, \mathbf{p}_T) \gamma^+ = \left\{ f_1(x, \mathbf{p}_T^2) + i h_1^\perp(x, \mathbf{p}_T^2) \frac{\not{p}_T}{M} \right\} \mathcal{P}_+ \quad (7)$$

$$\Phi_L(x, \mathbf{p}_T) \gamma^+ = \left\{ S_L g_{1L}(x, \mathbf{p}_T^2) \gamma_5 + S_L h_{1L}^\perp(x, \mathbf{p}_T^2) \gamma_5 \frac{\not{p}_T}{M} \right\} \mathcal{P}_+ \quad (8)$$

$$\begin{aligned} \Phi_T(x, \mathbf{p}_T) \gamma^+ = & \left\{ f_{1T}^\perp(x, \mathbf{p}_T^2) \frac{\epsilon_{T\rho\sigma} p_T^\rho S_T^\sigma}{M} + g_{1T}(x, \mathbf{p}_T^2) \frac{\mathbf{p}_T \cdot \mathbf{S}_T}{M} \gamma_5 \right. \\ & \left. + h_{1T}(x, \mathbf{p}_T^2) \gamma_5 \not{S}_T + h_{1T}^\perp(x, \mathbf{p}_T^2) \frac{\mathbf{p}_T \cdot \mathbf{S}_T}{M} \gamma_5 \frac{\not{p}_T}{M} \right\} \mathcal{P}_+. \end{aligned} \quad (9)$$

Choosing for the above objects a convenient (Weyl) representation, one sees that they are effectively  $2 \times 2$  in quark spin space. The leading part of the correlator is spanned by just two types of quarks; left and right-handed (good) quarks.

If we now put everything together to obtain  $\Phi(x, p_T)\gamma^+$  (or it's transpose in dirac space  $(\Phi\gamma^+)^T$ , to be more precise), from an expression like (5), one concludes that the  $\tilde{M}$  needed for the description including transverse momentum, is the following.

$$\begin{pmatrix} f_1 + g_{1L} & \begin{pmatrix} \frac{|p_T|}{M} e^{i\phi} \times \\ (g_{1T} + i f_{1T}^\perp) \end{pmatrix} & \begin{pmatrix} \frac{|p_T|}{M} e^{-i\phi} \times \\ (h_{1L}^\perp + i h_1^\perp) \end{pmatrix} & 2 h_1 \\ \begin{pmatrix} \frac{|p_T|}{M} e^{-i\phi} \times \\ (g_{1T} - i f_{1T}^\perp) \end{pmatrix} & f_1 - g_{1L} & \frac{|p_T|^2}{M^2} e^{-2i\phi} h_{1T}^\perp & \begin{pmatrix} -\frac{|p_T|}{M} e^{-i\phi} \times \\ (h_{1L}^\perp - i h_1^\perp) \end{pmatrix} \\ \begin{pmatrix} \frac{|p_T|}{M} e^{i\phi} \times \\ (h_{1L}^\perp - i h_1^\perp) \end{pmatrix} & \frac{|p_T|^2}{M^2} e^{2i\phi} h_{1T}^\perp & f_1 - g_{1L} & \begin{pmatrix} -\frac{|p_T|}{M} e^{i\phi} \times \\ (g_{1T} - i f_{1T}^\perp) \end{pmatrix} \\ 2 h_1 & \begin{pmatrix} -\frac{|p_T|}{M} e^{i\phi} \times \\ (h_{1L}^\perp + i h_1^\perp) \end{pmatrix} & \begin{pmatrix} -\frac{|p_T|}{M} e^{-i\phi} \times \\ (g_{1T} + i f_{1T}^\perp) \end{pmatrix} & f_1 + g_{1L} \end{pmatrix} \quad (10)$$

This matrix lives in the product space of target helicity and quark handedness. The upper-right as the lower-left  $2 \times 2$  submatrices are solely populated by so called chiral-odd functions [12, 13, 14], that involve the flipping of quark handedness, whereas the diagonal submatrices contain merely chiral-even functions. Whithin each of these  $2 \times 2$  matrices, the diagonal elements involve no and longitudinal polarization, as these states can be expressed in the helicity eigenstates of the target, whereas the non-diagonal ones involve transverse polarization as expected from (5). In (10) all distribution functions particular to transverse quark momentum are accompanied by an azimuthal dependence. This dependence averages to zero after integration over azimuthal angle, showing that taking into account transverse momentum is necessary to access the full helicity structure of a polarized nucleon [15]. The sought for bounds follow from the fact that for any vector  $a$  the quantity  $aMa \geq 0$ . If an integration over azimuthal angle is performed first and after that positive semi-definiteness is demanded, one finds the Soffer bound. Note that the T-odd functions  $f_{1T}^\perp$  and  $h_1^\perp$  can be considered as imaginary parts of  $g_{1T}$  and  $h_{1L}^\perp$ , respectively.

### 3 Interpreting the bounds

Regarding 2-dimensional subspaces in (10) starts giving us non-trivial bounds on the distribution functions. Omitting the  $(x, p_T^2)$  dependences of these functions one finds

$$|h_1| \leq \frac{1}{2} (f_1 + g_{1L}) \leq f_1, \quad (11)$$

$$\frac{p_T^2}{2M^2} |h_{1T}^\perp| \leq \frac{1}{2} (f_1 - g_{1L}) \leq f_1, \quad (12)$$

$$\frac{\mathbf{p}_T^4}{4M^4} \left( (g_{1T})^2 + (f_{1T}^\perp)^2 \right) \leq \frac{\mathbf{p}_T^2}{4M^2} (f_1 + g_{1L})(f_1 - g_{1L}) \leq \frac{\mathbf{p}_T^2}{4M^2} f_1^2, \quad (13)$$

$$\frac{\mathbf{p}_T^4}{4M^4} \left( (h_{1L}^\perp)^2 + (h_1^\perp)^2 \right) \leq \frac{\mathbf{p}_T^2}{4M^2} (f_1 + g_{1L})(f_1 - g_{1L}) \leq \frac{\mathbf{p}_T^2}{4M^2} f_1^2. \quad (14)$$

In order to incorporate the more elaborate bounds that are found considering higher dimensional subspaces of the matrix (10), it is convenient to introduce two positive-definite functions  $A(x, \mathbf{p}_T^2)$  and  $B(x, \mathbf{p}_T^2)$  such that  $f_1 = A + B$  and  $g_1 = A - B$  and also

$$h_1 = \alpha A, \quad (15)$$

$$\frac{\mathbf{p}_T^2}{2M^2} h_{1T}^\perp = \beta B, \quad (16)$$

$$\frac{\mathbf{p}_T^2}{2M^2} (g_{1T} + i f_{1T}^\perp) = \gamma \frac{|\mathbf{p}_T|}{M} \sqrt{AB}, \quad (17)$$

$$\frac{\mathbf{p}_T^2}{2M^2} (h_{1L}^\perp + i h_1^\perp) = \delta \frac{|\mathbf{p}_T|}{M} \sqrt{AB}. \quad (18)$$

Here  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  all depend on both  $x$  and  $\mathbf{p}_T^2$  and have absolute values in the interval  $[-1, 1]$ . Note that  $\alpha$  and  $\beta$  are real-valued whereas  $\gamma$  and  $\delta$  are complex-valued. Their imaginary parts determine the strength of the T-odd functions.

In terms of these functions, what is required to be positive semi-definite are the following four expressions

$$e_{1,2} = (1 - \alpha)A + (1 + \beta)B \pm \sqrt{4AB|\gamma + \delta|^2 + ((1 - \alpha)A - (1 + \beta)B)^2}, \quad (19)$$

$$e_{3,4} = (1 + \alpha)A + (1 - \beta)B \pm \sqrt{4AB|\gamma - \delta|^2 + ((1 + \alpha)A - (1 - \beta)B)^2}, \quad (20)$$

leading to

$$A + B \geq 0, \quad (21)$$

$$|\alpha A - \beta B| \leq A + B, \quad \text{i.e. } |h_{1T}| \leq f_1, \quad (22)$$

$$|\gamma + \delta|^2 \leq (1 - \alpha)(1 + \beta), \quad (23)$$

$$|\gamma - \delta|^2 \leq (1 + \alpha)(1 - \beta). \quad (24)$$

In figure 1 one can see the a graphical representation of the allowed values for  $\alpha$  and  $\beta$ . It is remarkable to see that to see that an inclusively measured function as  $h_1$  is involved in a bound including functions as  $g_{1T}$  and  $h_{1L}^\perp$  which cannot be measured inclusively and are responsible for asymmetries [8, 16].

#### 4 Concluding remarks

It is important to note that though in this talk only distribution functions have been addressed, an almost identical analysis can be performed on fragmentation functions [17]. The non-vanishing of T-odd functions, though disputed in the case of distribution functions yet a possibility [18], is accepted in the case of fragmentation

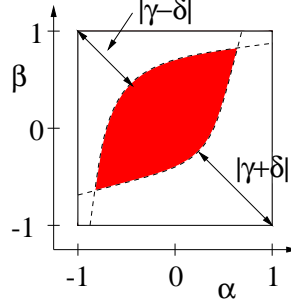


Fig. 1. Allowed region (shaded) for  $\alpha$  and  $\beta$  depending on  $\gamma$  and  $\delta$ .

functions, such as  $D_{1T}^\perp$  [8, 19] and  $H_1^\perp$  [20], as time-reversal invariance cannot be imposed on the final state [21, 22, 23]. This work can also straightforwardly be extended to spin-1 hadrons [24] and gluons [25]. Though it should be supplemented with a study of the factorization, scheme dependence and stability of the bounds under  $Q^2$  evolution [26], these bounds provide an estimate of the magnitudes of functions measured in asymmetries at SMC [27], HERMES [28] and LEP [29].

### References

- [1] A.Bacchetta, M.Boglione, A.Henneman and P.J. Mulders, Phys. Rev. Lett. 85, (2000) 712.
- [2] D.E. Soper, Phys. Rev. D 15 (1977) 1141; Phys. Rev. Lett. 43 (1979) 1847.
- [3] R.L. Jaffe, Nucl. Phys. B 229 (1983) 205.
- [4] A.V. Manohar, Phys. Rev. Lett. 65 (1990) 2511.
- [5] Other common notations for a quark flavor  $q$  are  $f_1^q(x) = q(x)$ ,  $g_1^q(x) = \Delta q(x)$  and  $h_1^q(x) = \Delta_T q(x)$ .
- [6] J.B. Kogut and D.E. Soper, Phys. Rev. D 1 (1970) 2901.
- [7] J. Soffer, Phys. Rev. Lett. 74 (1995) 1292.
- [8] P.J. Mulders and R.D. Tangerman, Nucl. Phys. B 461 (1996) 197; Nucl. Phys. B 484 (1997) 538 (E).
- [9] R. D. Tangerman and P.J. Mulders, Phys. Rev. D 51 (1995) 3357.
- [10] J.P. Ralston and D.E. Soper, Nucl. Phys. B 152 (1979) 109.
- [11] D.Boer and P.J. Mulders, Phys.Rev. D 57 (1998) 5780.; J.Levelt and P.J. Mulders, Phys. Lett. B 338 (1994) 357.
- [12] X. Artru and M. Mekhfi, Z. Phys. 45 (1990) 669.
- [13] J.L. Cortes, B. Pire and J.P. Ralston, Z. Phys. C 55 (1992) 409.
- [14] R.L. Jaffe and X. Ji, Nucl. Phys. B 375 (1992) 527.
- [15] M. Boglione and P.J. Mulders, Phys. Rev. D 60 (1999) 054007.
- [16] D. Boer and P.J. Mulders, Phys. Rev. D 57 (1998) 5780.

- [17] J.C. Collins and D.E. Soper, Nucl. Phys. B 194 (1982) 445.
- [18] Possible T-odd effects could arise from soft initial state interactions as outlined in D. Sivers, Phys. Rev. D 41 (1990) 83 and Phys. Rev. D 43 (1991) 261. Also gluonic poles might lead to presence of T-odd functions, see N. Hammon, O. Teryaev and A. Schäfer, Phys. Lett. B 390 (1997) 409 and D. Boer, P.J. Mulders and O.V. Teryaev, Phys. Rev. D 57 (1998) 3057.
- [19] R.L. Jaffe, Phys. Rev. D 54 (1996) 6581. Here the notation  $\hat{f}_{...}$ ,  $\hat{g}_{...}$ ,  $\hat{h}_{...}$ , is used instead of  $D_{...}$ ,  $G_{...}$ ,  $H_{...}$ .
- [20] J. Collins, Nucl. Phys. B 396 (1993) 161.
- [21] A. De Rújula, J.M. Kaplan and E. de Rafael, Nucl. Phys. B 35 (1971) 365.
- [22] K. Hagiwara, K. Hikasa and N. Kai, Phys. Rev. D 27 (1983) 84.
- [23] R.L. Jaffe and X. Ji, Phys. Rev. Lett. 71 (1993) 2547.
- [24] A. Bacchetta and P.J. Mulders, hep-ph/0007120.
- [25] P.J. Mulders and J.Rodrigues, hep-ph/0009343.
- [26] C. Bourrely, J. Soffer and O.V. Teryaev, Phys. Lett. B 420 (1998) 375; G. Altarelli, S. Forte and G. Ridolfi, Nucl. Phys. B 534 (1998) 277.
- [27] A. Bravar, Nucl. Phys. Proc. Suppl. B 79 (1999) 521.
- [28] H. Avakian, Nucl. Phys. Proc. Suppl. B 79 (1999) 523.
- [29] E. Efremov, O.G. Smirnova and L.G. Tkatchev, Nucl. Phys. Proc. Suppl. B 79 (1999) 554.